Embodied objects and the signs of mathematics

Gary E. Davis
Graduate School of Education
Rutgers University
10 Seminary Place
New Brunswick, NJ 08901-1108 USA

Mercedes A. McGowen
Department of Mathematical Sciences
William Rainey Harper College
1200 West Algonquin Road
Palatine, IL 60067-7398 USA

E-mail: mmcgowen@harper.cc.il.us

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Introduction

We consider Gray & Tall's (2001) idea of embodied objects in mathematics from a semiotic perspective. We explore two main issues arising from Gray & Tall’s notion of embodied objects in mathematics. The first is that the simplest embodied objects of a mathematical type, such as triangles, are fundamentally different objects than the everyday objects such as cows, hamburgers and paper bags. The everyday objects are instances of perceptual categorizations, while the more mathematical objects also serve a referential, or communicative, function: they are marks made by humans to refer to something. This is not to say that these marks are necessarily, or even generally, interpreted this way: only that the maker of the marks had a referential intent in making the marks. Generally, we do not ascribe a referential or communicative function to a cow or a hamburger (although a representation of a hamburger, on a diner advertisement, for example, would serve a communicative purpose).

The second aspect of more mathematical embodied objects that we explore is their level in Peirce’s semiotic hierarchy (Deacon, 1997). What is commonly thought of as the “language” of mathematics consists of marks and drawings, and stylized diagrams, together with common and idiosyncratic language terms, all of which are intended to convey something from one person to another. From this viewpoint the “language” of mathematics begs to be interpreted semiotically. This thought is not original with us, and has been pursued by Becker & Varelas (1997), Lemke (2001), Vile (1997a, b) and Vile & Lerman (1996) among others. We are interested in how, beginning with the notion of embodied object, we can trace individual semiotic development through the marks and diagrams of mathematics. A major point for us is that simple embodied mathematical objects are, for many people, at the iconic level, the lowest of Peirce’s three major levels - the others being indexical and symbolic (see, for example, Buchler, 1955; Deacon, 1997). This is in sharp contrast to mathematical entities such as algebraic expressions, which also serve a referential, communicative function, but are at the very least at the next level of the semiotic hierarchy. They are for the vast majority of students indexical at least, and symbolic at best.

This, we argue, is the reason for the considerable conceptual split observed by Gray and Tall (2001) between embodied objects and more syntactical entities, such as algebraic expressions. The proceptual divide, formulated and described by Gray & Tall (1994, 2001), appears as a consequence of the fact that Pierce’s semiotic distinctions—iconic, indexical, symbolic—form hierarchical levels. That is, the proceptual divide is not a fundamentally new phenomenon—it is
the manifestation in a mathematical domain of a basic semiotic hierarchy. To understand the
dynamics of the proceptual divide—its formation, change, and stability—it is helpful, therefore,
to understand the transition from one semiotic level to another. Such understandings are based in
the changing patterns of our brains as we interact with the world, and our growth in ability to uti-
lize symbolic thought. Deacon (1997) outlines neurological evidence and models for these transi-
tions. In essence, we argue that the proceptual divide arises because there is a fundamental
distinction in various communicative interpretations of signs and marks in mathematical settings.
This distinction, fundamental to the semiotic hierarchy, is based on differing brain activities, par-
ticularly those related to memory, as students engage in mathematical activities.

**Embodied objects**

Ideas about the embodiment of thought have received a lot of attention in recent years
(see, for example, Dreyfus, 1982, 1996; Johnson, 1987; Varela, Thompson & Rosch, 1991; Lakoff
& Johnson, 1999; Lakoff & Núñez, 2000; Seitz, 2000). Many of these ideas can be traced to Mat-
urana and Varela’s writings on the biology of human cognition (Maturana & Varela, 1980, 1998)
or to Merlau-Ponty’s work on the embodiment of perception and cognition (Merlau-Ponty, 1962,
1968). Building on the ideas of Lakoff & Núñez, among others, Gray & Tall (2001) propose a
notion of embodied objects in mathematics. They do this partly to distinguish between plainly
object-like things in mathematics such as triangles and lines (geometric objects) on the one hand
and less obviously object-like things such as numbers, algebraic expressions, and algebraic struc-
tures. They allude, through a series of examples, to the simplest of embodied objects. These
include such everyday objects as cows, hamburgers and paper bags, and more clearly mathemati-
cal objects such as triangles, dots on a domino, graphs of functions, and Venn diagrams. They use
embodied objects to draw a distinction between the object-like things of mathematics, such as
numbers and algebraic expressions on the one hand, and embodied objects such as an image of
fingers used for counting on the other. This distinction goes some way to resolving Dörfler’s
objection to the number “5” as a mathematical object (Dörfler, 1993). Gray & Tall claim that it is
a mathematical object: it simply is not an embodied object.

We understand the notion of “embodied object”, in its simplest form, to mean a long term
declarative memory resulting from sensory perception, actions on physical objects, categorization
of those perceptions and actions, and personal values associated with the actions. Thus, an
embodied object of the simplest type generally has the character of a recalled image, recalled
actions on a physical object, and recalled feelings about those actions. Pick up an egg, feel it, roll it along a bench top and see how it wobbles, place it in water and see whether it floats or sinks, throw it and see if it cracks. Recall doing this and recall the feelings as you carried out these actions. These make the egg more than just a categorized object — it is now an object acted upon, and recalled through values, associated with surprise, shock, and satisfaction. Now do the same with a white plastic model of an egg: the recollections are different, though the perceptual form of both physical objects is similar in appearance and feel. An “embodied object” of the simplest type begins with mental conceptions of a physical object perceived through the senses, together with reflection on what is perceived, how it is acted on, and comparison with other objects - physical or embodied - and shared with, or communicated to, others. More mathematical examples of simple embodied objects consist of memories of three pieces of wood, pinned at the ends to form a triangular shape, and four pieces of wood pinned together at their ends to form a square shape.

**FIGURE 1:** Diagrams intended to resemble two shapes made from wooden pieces pinned at their ends.

Visually the two shapes appear to be different. Counting establishes one has 3 sides, the other 4. Pushing and pulling reveals that one shape is rigid and the other is not. Forming long-term memories of these perceptions and actions, and recalling them with values, constitute examples of simple embodied objects with a mathematical flavor, stemming from actions of counting, pushing and pulling. In other words, embodied objects are memories that result from the transformation of episodic memories, formed as a result of physical actions on concrete objects, into long-term semantic memories.

**Icons**

What makes written and spoken words, and written diagrams and expressions in mathematics, more than marks in the sand, or on paper, or a computer screen, or simply vibrations in the air, is the interpretations we place on them. It is no surprise that different people, at different times, place differing interpretation on various words, signs, marks, and sounds. Teachers of mathematics generally want students to place interpretations on signs, including expressions and
diagrams, that are similar to, if not the same as, the teacher’s. But what are those interpretations, and how do the signs and expressions of mathematics convey the communicative intent of a teacher?

The simplest of three classes of signs in Peirce’s semiotics is the class of icons, or likenesses. When a collections of marks or sounds is iconic for someone, it is so because it brings to mind for that person something else, which it resembles. The following marks are iconic for most people:

![Figure 2: Two drawings widely regarded as iconic for (a) a human face, and (b) a clock face](image)

Mathematical examples of icons are not as common as one might think. Drawings such as the following can fairly safely be assumed to be iconic for many people:

![Figure 3: (a) A diagram that is iconic, for the authors, for a function machine. (b) A diagram that is potentially iconic, for a sphere, or for a disk.](image)

A triangle is a classic geometric object. This is one of Gray & Tall’s (2001) examples of a simple embodied (mathematical) object. Nothing in mathematics, it seems, could be more iconic than a drawing of a triangle:

![Figure 4: A drawing of a triangle. To what does it refer?](image)
However, if a drawing of a triangle is iconic for something, then what might that something be? The decisive question is this: If a triangle is a likeness, of what is it a likeness? Pieces of wood as in figure 1? A triangle? But what, exactly, is a triangle? We can answer this in many ways, dependent on our level of geometrical experience. One answer, related to a triangle as an embodied object, is that the drawing is iconic for those sorts of things that builders and architects use in buildings to help keep them rigid: the sort of physical things indicated in figure 1. Another interpretation, not quite so clearly iconic, is that the drawing indicates a path taken by an airplane flying from A to B to C and then back to A again, although to be more plainly iconic in this context we, the authors, would prefer to see the triangles drawn more like the following:

![Figure 5: A drawing meant to resemble a triangular path on a sphere.](image)

These interpretations of marks called a “triangle” are iconic in that the marks resemble an object in the world external to the body of the interpreter. However, many school children are introduced to triangles as the marks in mathematics books that are labelled “triangle”. For these students the marks in figure 4 are iconic for what they have seen in their text books—they resemble nothing so much as the marks teachers draw when talking about geometry. There is simply no universal answer for what a generally accepted icon is iconic for: interpretations can, and do, vary. Another example where a mark that is iconic for many students of advanced mathematics stands for an embodied object, is in a drawing of a two-dimensional torus:

![Figure 6: (a) A drawing of a torus, (b) A drawing indicating how a torus can be obtained by gluing opposite ends of a flexible tube.](image)

For what might figure 6(a) be iconic? Our answer is that it is most likely, for most people, to be iconic for a physical object obtained from a piece of flexible tubing by gluing opposite ends.
This drawing is misleading as far as the geometry of the torus is concerned. It makes the torus look curved, whereas the geometry of the torus is inherited from a flat rectangle and has zero curvature (Thurston, 1997). It is not what geometers understand by a geometric torus: the implied curved geometry, inherited from 3-dimensional space, is not the intrinsic flat geometry of the torus. Insofar as the figure is iconic it is highly likely to be an icon for a thing not intended by a geometer who drew it. A torus, as shown in figure 6, is an embodied object in the sense of Gray & Tall. However, its interpretation—to be consistent with the thinking of geometers—requires a level of reference higher than the iconic in order to be interpreted consistently at the geometric level where distances between points matter. This is an example where the apparent iconicity of a drawing is misleading in an important geometric aspect. There are many others in geometry. The nature of icons, and of embodied objects in mathematics, particularly in geometry, encompasses many subtleties, and appearances are not always what they seem at first blush.

Two- and three-dimensional vectors are candidates for iconic embodied objects. A vector as an embodied object is drawn as an arrow. Many such vector drawings can be fitted together to give a drawing of a vector field, a highly iconic embodied object of advanced mathematical thinking:

![Figure 7: A plane vector field](image)

This resembles a flow of a liquid, or flow lines of some other field, in a plane. In that sense the drawing is iconic. This, by the way, was not the authors’ intention in producing this picture. It was made for a different reason and has, for the authors, a different interpretation. The picture represents for them a vector field that shows the eigenspaces of a linear transformation of the plane. It was produced to illustrate an “object” to which the words “eigenvalue” and “eigenspace” pointed. A single vector depicted as an arrow is much less obviously iconic—a drawing such as that in fig-
ure 8, below, is, to our minds, not likely to be interpreted iconically, in a mathematical context, as a vector: what does it resemble?

**FIGURE 8: A free vector**

Rather, as we shall see in the next section on indexes, a drawing such as that in figure 8 is much more likely to be interpreted indexically, via a conditioned response. It might be interpreted as an icon in that it resembles an arrow, but that is only a hook, a visual distraction, to the intended reference for a vector. The main reference for such diagrams is intended to point to the length and direction of vectors, and to their addition and scalar multiplication, not to physical arrows or spears.

Marks, drawings, and sounds are not in and of themselves iconic (Deacon, 1997). The iconicity of an icon does not reside in the physical marks: it resides in the level of interpretation placed on those marks by someone paying attention to them. There are no “triangles” in the world: only different interpretations of “triangle-like things” in our minds. Put simply, a person who tells you they know what a triangle is, is severely limiting their possible or potential interpretations. They know what they know, but being hard pressed to say, for example, what a “line” is they will be even more hard pressed to explain triangles.

The reaction, or response, to a mark such as figure 4 is a semiotic one: an act of interpretation. A human made the mark, with a certain intent. Persons interpreting the mark do so in their own way, dependent both on their own experience and on their unique level of semiotic development. One might argue that figure 4 is not a drawing “of” anything: it simply is what it is. Indeed, without knowing the intent of the drawer of figure 4 we cannot know if the figure was intended to point to something else. However, we do know that in most mathematics books and drawings of triangles in classrooms, the author of those drawings does not simply intend the drawing to be a thing in itself. Those drawings definitely point to something else, even if that something else is only other similar looking drawings. A teacher of mathematics rarely draws a triangle to reflect on aspects of that particular drawing. The example of a triangle as “embodied object” highlights the fundamental role of reference in the signs of mathematics (as in everyday language and communication). The “triangularity” of a triangle does not reside in the drawing of figure 4. It lies in the
reference intended by the maker of that diagram, and in the references and associations attributed to it by a perceiver of the diagram.

**Indexes**

An index is a sign that refers to something else by association. Examples commonly known to many people are the sounding of a bell at the start of the school day, the sound of an alarm clock in the morning, a pointer on a car’s gas gauge, and a painting of a skull and crossbones on a road. An important aspect of indexes, fundamental to their position in Peirce’s semiotic hierarchy, is that they organize icons. Three iconic relationships are involved in a sign whose reference is interpreted indexically (Deacon, 1997, p. 79). It is as if an indexical relationship were an arrow with the tail as one icon—the stimulus—the tip as another icon—the reference—and the body of the arrow another icon—the correlation of the other icons in time or space:

![Stimulus icon](Stimulus_icon.png) ![Reference icon](Reference_icon.png)

**FIGURE 9:** A schematic of an index as arrow, viewed as three organized icons.

A simple everyday example is the sound of scraping food from a can for a cat. The sound of the scraping is a stimulus for the cat which, though not identical with, resembles other sounds the cat has heard before. This particular scraping, therefore, is iconic for “scraping” in general: it does not take much interpretation, and is like other remembered instances of scraping. The cat remembers something else: food. Not always the same food, but always something being edible. This is a second iconic memory for the cat, brought to mind as a consequence of the scraping noise. Finally, this situation is similar to (resembles, is iconic with) other past situations in which scraping and food have gone together. Thus the scraping sound as an index for the cat involves an aural icon, a visual and olfactory icon, and a temporal correlational icon in which the other two icons are treated, in memory, as part of a whole. It is in this sense that the sound of scraping acts as an indexical pointer for food.

A mathematical example of an index might be a quadratic equation for a high school student. The student might not be able to define or explain in words exactly what a quadratic equation is, but the one he sees now, in front of him, resembles other things he recalls being called “quadratic”. In the past these things have been correlated with working out sums and finding
“solutions”. The solutions are not always the same thing, but they resemble each other in so far as there are calculations to be done and they always come out as “\(x_1 = \ldots, x_2 = \ldots\)”. So, like the cat, the student has three iconic references integrated into an index. These icons are: the similarity of the expression in front of him to other expressions, the similarity of the solution process to other solutions, and the memory of the correlation of these two things. In this sense the quadratic equation the student sees points to, or refers to, the solutions. The signs that constitute the quadratic equation are interpreted by the student in terms of memories of solution processes.

**Symbols**

Symbols, unlike icons and indexes, are not explainable as single items. As indexes organize icons in higher-order relationships, so symbols organize indexes in higher-order relationships. There are two essential differences however. A sign cannot be a symbol in and of itself—it must be involved in conventional relationships with other signs that also participate in a system of symbols. Thus symbols, unlike icons and indexes, do not stand alone: they form part of a connected symbolic system. Second, a system of symbols is not learned as indexes are learned from experience and memory. A system of symbols has to be apprehended, pretty much all at once. The symbolic associations in signs are understood from the myriad connections between indexes. Deacon (1997) has a simple, yet descriptive, way of delineating the difference between what any given person sees as iconic, indexical, and symbolic relationships:

“… the differences between iconic, indexical, and symbolic relationships derive from regarding things either with respect to their form, their correlations with other things, or their involvement in systems of conventional relationships.” (p.71)

“… the shift from associative predictions to symbolic predictions is initially a change in mnemonic strategy, a recoding. It is a way of off-loading redundant details from working memory, by recognizing a higher order regularity in the mess of associations, a trick that can accomplish the same task without having to hold all the details in mind. “(p. 89)

The latter is reminiscent of Barnard & Tall’s (1997) use of the phrase *cognitive unit*—”a piece of cognitive structure that can be held in the focus of attention all at one time”. However, this is only part of the idea of a symbol in the sense of Peirce, and the following quote from Crowley & Tall (1999) shows that Tall & co-workers have not (yet) placed the idea of a cognitive unit in the semiotic context of symbol:
“What is important to be able to compress a collection of related ideas into a cognitive unit is that the whole entity can be conceived as a unit that is “small enough” to be considered consciously, all at one time. The way that the human mind usually copes with this is to give it a name or symbol. The name or symbol (assuming it is “small enough”) can be held in the focus of attention and manipulated. Such a concept has rich interiority through carrying “within” it many powerful links that enable it to be manipulated and invoked to solve problems. “

McGowen (1998) elaborates Barnard & Tall’s use of cognitive unit as “those bits and pieces of knowledge chunked together that can be held in the focus of attention (i.e. held in working memory), which acts as the cue for retrieval and selection of the schema which determine subsequent actions or those facets of a concept image needed for the task at hand. “(p. 61). This formulation of cognitive unit emphasizes not only the compressed nature of a cognitive unit, and its consequent role in relieving working memory, but also its cueing nature—what it points to in memory.

An important point about a symbol is that it not only can be “held in the focus of attention and manipulated” but that it is linked, through a process of paying attention to higher-order associations, with other symbols:

“Although the prior associations that will eventually be recoded into a symbolic system may take considerable time and effort to learn, the symbolic recoding of these relationships is not learned in the same way; it must instead be discovered or perceived in some sense, by reflecting on what is already known. In other words it is an implicit pattern that must recognized in the relationships between the indexical associations. “(Deacon, 1997, p. 93)

To return to our discussion of triangles, were we studying Galois fields and working with finite geometries extensively, we might see the diagram in figure 4 not as iconic or simply as an index but as a symbol referring to something like the following (ref. Beck et al., 2000, p. 301) in a network of symbolic references:

![FIGURE 10: A triangle in the finite plane GF(4) x CF(4): the product of two copies of the field of 4 elements— with vertices shown as black.](image)
This is a more advanced form of reference than an index, in that the knowledge required to form the correlations involved in the interpretation require more advanced study, most of it symbolic.

We sketch below an example of a transition to symbolic associations from our recent experience with pre-service elementary teachers. (Davis & McGowen, 2001; McGowen & Davis, 2001; McGowen & Davis, 2001). In the Fall of 2000, 19 pre-service teachers worked at Harper College, in groups, for the first 3 weeks of a 16 week semester, on the following problems:

- How many towers of height 4 can be made from blocks of 2 colors? How many towers of height 5?
- On the grid shown below, indicate on each dot how many different ways there are to walk to that dot from “home” given that you can only walk UP or RIGHT.

![Figure 11: Portion of an integer lattice on which students performed walks UP or RIGHT from “home” to adjoining dots.](image)

- How many different ways are there to run through a series of 4 tunnels if you must pass through exactly two white and two black tunnels? See figure 12, below:

![Figure 12: At each point, the runner goes through only a black or a white tunnel.](image)

- \((a + b)^2 = a^2 + 2ab + b^2\). What are the expansions for \((a + b)^3\) and \((a + b)^4\)?
- What are the connections between towers, grid walks, tunnels and binomial expansions?

The pre-service teachers did not solve these problems by remembered formulas—the problems were chosen with that likelihood in mind. For example, in relation to the tower building problem, one student wrote:
“Instead of looking at it as a math problem, I was looking at it as a building exercise. I first attempted the problem by guessing and testing. (We) attempted the problem of four high by creating combinations of four that would design an obvious pattern.”

With one exception, the pre-service teachers could not expand the binomial expressions $(a + b)^2$ and $(a + b)^3$. The problem of discovering and articulating connections between the other problems stimulated one student to speculate about what those connections might be. He gave an explanation prefaced by a statement along the lines “this might be crazy, but…. $(a + b)^2 = a^2 + 2ab + b^2$“ could be interpreted in terms of block towers. He interpreted $a$ and $b$ as two colors, and the binomial expansion $(a + b)^2 = a^2 + 2ab + b^2$ as saying there is 1 tower of height two built from the “$a$” color, 1 from the “$b$” color, and 2 using both the “$a$” and “$b$” colors. He then used black ($B$) and white ($W$) and wrote the binomial expansion as $(B + W)^2 = B^2 + 2BW + W^2$. He interpreted the same expression in terms of grid walks by taking $B$ to mean “walk UP” and $W$ to mean “walk RIGHT”. Finally, he related towers and grid walks to the other binomial expansions, and the patterns of numbers on the grid to Pascal’s triangle.

This episode was etched into the memory of many of the students, as student write-ups and reflections show:

“If you go through the tunnel with the pattern of $BWBW$, you can also make a tower of $BWBW$. If you think of $B$ as equalling right $R$ and $W$ as equalling up $U$, you can go $RURU$. Algebraically, you can replace the $a$ and $b$ with a $B$ and $W$ and have $(B + W)$ raised to whichever high, which I’m using 4. It shows $(B + W)^4 = B^4 + 4B^3W + 6B^2W^2 + 4BW^3 + W^4$. If you add up all the numbers in front, you get the number of possibilities. The exponents are the number of the different colors you have. Ex: $4B^3W$ means there are 3 blacks and 1 white, with 4 different ways of arranging it. To have two of each like the tunnels ask, would be $6B^2W^2$, which means there are 6 possibilities with 2 black and 2 white.” (Reflection, Week 4)

“… although I used Pascal’s triangle many times throughout high school, I never truly understood why it worked. By completing the grid walk exercises, so many things have become clear to me. Well, I learned the secret of the grid walks and of Pascal’s Triangle. The two numbers directly above any given number represent the number of ways to get to that certain dot. I really wish I had known this when I was in Algebra. All we were taught then was that, for some reason unknown to me and probably many others in the class, the rows of Pascal’s Triangle made up the coefficients in binomial expansion. Now I not only understand how to use the triangle algebraically, I know why it works. The numbers in an algebraic expression along with the variables can represent different groups of objects, in a manner of speak-
ing. It all makes sense. This simple understanding, but by no means easy, is so rewarding because the knowledge has become my own. And I know that ten years from now, I will still be able to explain this problem.” (End-of-semester write-up, week 16)

Many—but not all—of the pre-service teachers used these interpretations of the syntactic expansions of the binomial expressions in the solution of other problems. For example, in solving how many pizzas can be made from a base and 8 possible toppings, some students responded by seeing the pizza problem as representing two choices—each topping either being present or absent:

“\( n = \) without toppings; \( w = \) with toppings; \( (n + w)^8 = \) total combinations;
coefficients = number in each of the subsets. Subsets = binomial coefficients”

The semantic references for the binomial expressions \( (a + b)^n \) changed. Formerly, the references were at best indexical—a conditioned response signifying nothing more than a way of acting in a remembered context. The references for the binomial expressions now become symbolic, referring both to manifold objects and processes, as well as to other syntactic marks, such as \( 6a^2b^2 \). This, was not so much a learning task, in the sense of learning a skill, or learning to respond, as a discovery task. The students’ task, set by us, was to discover higher-order associations in the marks and signs, not just those associations that were indexical. We could have simply “told” the students what to do, but what would that have achieved? Would their thinking about binomial expressions have become symbolic if we told them the connections between towers, grid walks, tunnels and binomials that we saw? We doubt it. Most likely they would have “learned” these facts as they would learn any other conventional fact, committing them to memory and retrieving them under appropriate circumstance. In other words, had we told them of the connections, they would, in all likelihood, have seen these connections indexically. By discovering the higher-order associations, through a process of guided discovery, a process charged with significant emotional and intellectual energy and effort, they had the possibility of seeing symbolic connections. For those who did, a veil was lifted, and suddenly—not gradually—they “saw”.

For those pre-service teachers who made the symbolic transition, algebra in the form of exponential functions and binomial expansions was no longer a series of marks signifying only that they should do something (“simply just plugging numbers into a learned equation”, for example). Algebra and algebraic language now became a means of expressing relationships between
objects and processes—a way of talking about and describing experiences in pattern and arrangement. The algebraic formulas, such as \((a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\), developed meaning via manifold references.

Learning to operate symbolically in advanced mathematics is not about learning to manipulate mathematical squiggles. That is a manipulation task—a series of sophisticated indexical responses. Operating symbolically means focusing on something other than the indexical associations: it means discovering higher-order associations in the mass of indexical associations. One could see this as a transition to “understanding”.

It is the recoding of indexical relationships and the embedding of a putative symbol in a system of symbols that gives rise to the genuinely symbolic nature of a token. This is an essential insight that semiotics brings to the study of symbols. Another is the assertion that the three semiotic levels—iconic, indexical, and symbolic—are hierarchical: indexes organize icons, and symbols organize indexes. We consider in the final section of this article the idea that deductive mathematics—a crucial aspect of advanced mathematical thinking—involves organizing symbols through logical necessity. We hypothesize that this constitutes a higher semiotic level than the symbolic as it is understood in the use of everyday language.

An essential feature of human marks and signs is that they are not intrinsically iconic, indexical or symbolic. That is, the semiotic level of a sign does not reside simply in the physical sign: it requires an act of interpretation by a person whose attention is directed to the sign. We can only speak, therefore, of signs being icons, indices, or symbols, for a particular person, at a particular time, in a given context. These contingencies might seem to render the notion of the semiotic level of a sign useless: what can we gain from studying something so context dependent? This is a good question, and our answer to it lies in the nature of student development in mathematics. We are interested in students’ increasing semiotic sophistication in interpreting the marks and signs of mathematics. We want to know whether, and how, students are progressing from one semiotic level to another.

**Hearing, using, and becoming familiar with, mathematical words**

Symbolic associations are socially mediated. Communication is a driving force in the establishment of symbolic insights. To this end, the use by teachers and students of mathematical words and phrases is, in all likelihood, an essential precursor for the development of full-blown
symbolic associations. In the excerpt below we see how a pre-service elementary teacher has changed her field of reference for the conventional number word “ten”:

“The most important thing I learned this semester so far is how important relational learning is and also that visualizing the work being done in mathematics is also very important and necessary for understanding concepts. I’ve learned a lot about the role of place value when we did that whole section on bits, longs, and flats. When we first started playing with them I thought what we were doing was crazy, and I didn’t see the connection of the bits to our number system. The more we talked about it the more I understood.... I have to admit that, since our number system is in the base of ten, when I think of the number ten, ten bits come to mind.” (Our italics)

This student goes on to support Gray & Tall’s notion of embodied objects in mathematics through the use of names for mathematical objects:

“This reminds me of what I read about relational understanding in Skemp’s article. It makes more sense to look at an object representing an item first, then attaching the name to it in your head. You can’t know what an apple is unless you’ve seen it and touched it. “

Our notion of a simple embodied object is based on long term declarative memory, and on semantic, as distinct from episodic, memory in particular. These semantic memories are known “facts” about mathematics—what a student recalls as being true (often referred to as semantic facts—what is “known”, in everyday language, versus what is “remembered”; see Gardiner & Richardson-Klavehn, 2000). However, mathematics, in our experience, admits various forms of semantic memory, so embodied objects are potentially of different types. There are, at least, the following distinctions in memory for mathematical facts:

1. Labels, customs, and conventions. For example: A prime number is a whole number with exactly 2 factors.

2. Things sensed, or done. For example: The proportion of prime numbers less than 500 is 19%.

3. Things believed. For example: There are infinitely many prime numbers.

4. Things explained. For example: A proof that there are infinitely many prime numbers.

We refer to these remembered facts as semantic conventions, semantic actions, semantic beliefs, and semantic explanations, respectively. Episodic memories and semantic actions—memories of things sensed or done in mathematical settings—are easily confounded. The reason, of course, is that a semantic action, by its very nature, involves memory of sensing or doing some-
thing. However, the critical difference is that one senses or does some non-trivial fact—calculating the number of primes less than 100, for example. The episode, remembered as such, is overlaid with an act or sense of a fact that goes beyond a mere episode. The distinction between episodic memories and semantic actions is one that distinguishes memory for mathematics from more everyday memories, as does, in part explanatory memory—memory of mathematical explanations.

Semantic conventions—the linguistic conventions of mathematics—are the simplest form of mathematical factual memory: the memories carrying least mathematical information. They are non the less interesting for that, because semantic conventions, i.e., declarative memories of conventional mathematical language, are indicators of the degree of engagement a student has with mathematics content. Typical semantic conventions relating to mathematics in the pre-service teachers’, Harper College, Fall 200, written work include:

- “Prime numbers are counting numbers with exactly and only two different factors.”
- “A bit is the smallest unit of measurement.”
- “Factors are numbers that can be multiplied together to get another number.”
- “Triangular numbers are the cumulative sum of counting numbers.”
- “… I discovered how each relates. Both present two choices, or a ‘binomial’.”
- “In our class discussion, we defined algorithm as a systematic procedure that one follows to find the answer.”
- “An algorithm is a step by step procedure to find an answer usually to a computation.”
- “A decimal is the dot that tells how large a number is based on where it is.”
- “I’ve learned a lot about placed value from decimal operations. A decimal is the dot that tells how large a number is based on where it is. Decimals increase from right to left by the multiplication of ten. Adding and subtracting decimals is pretty straightforward, just line them up and add or subtract. It’s just like adding or subtracting whole numbers.”

We can view these as “definitions”: the students’ understanding of conventional mathematical terms. Another perspective is to see the student use of these conventional terms as the beginnings of a symbolic referential system. The conventional terms, such as “prime number”, are embedded in a system of pre-existing language symbols that contains other mathematics specific terms, such as “factor”. Very much like reading a dictionary definition of a word or phrase and then both using it and explaining it in context, these students are embedding conventional mathe-
matical terms into their pre-existing linguistic symbolic systems. They are learning, in an indexical sense, the definitions of mathematical things such as prime numbers and algorithms. But this learning is equally a process of discovery of how the conventional term “prime number” becomes embedded as a symbol in a system of symbols. From this perspective, the ability to declare the meaning of conventional mathematical terms indicates a student’s mental re-organization from acting implicitly—working out prime numbers, for example—to being able to deal symbolically with the term “prime number”. This is a major mental re-organization for a student, requiring energy on their part to form explicit long-term memories and also to embed formerly indexical terms into a pre-existing symbolic system. This partly explains why learning definitions is so difficult. Rote recollection of the terms of the definition in an indexical manner is to focus on an unproductive, unintended aspect of the definition. Note, too that a student who uses the term “prime number” in a symbolic sense now has, according to our definition of embodied object, a concept of “prime number” as an embodied object.

The semiotic hierarchy and the proceptual divide

The phenomenon of the duality and ambiguity of mathematical notation perceived as procedure and concept has been proposed by Gray and Tall (1991) as an explanation of an underlying cause of elementary-grade students’ success or lack of success in mathematics. Subsequently, Gray and Tall (1994) hypothesized that the ability to think flexibly in mathematics depends on the dual use of mathematical signs for both procedure and concept, a duality found throughout mathematics. They defined the amalgam of procedure and concept which is represented by the same notation to be a procept. They use this notion to explain the divergence and qualitatively different kind of mathematical thought evidenced by more able thinkers compared to the less able. The sign “–3” is an example of a procept which can be interpreted in several ways, depending upon the context. If arithmetic operations are analyzed using the notion of function, –3 could be interpreted as either the unary process of taking the additive inverse (a process requiring one input) or as a mathematical object, the concept negative three (McGowen, 1998, pp. 104-105).

Krutetskii (1969) argues that an ability to think flexibly is an essential component of success in mathematics. Theories of ‘encapsulation’ focus on the manner in which processes are encapsulated as objects, which generally lead to quantifiable differences in procedures. Qualitative differences in more able students’ abilities to think successfully compared with the abilities of
less able students have been documented in the studies of Krutetskii (1969) and other Russian researchers including Dubrovina (1992) and Shapiro (1992).

The divergence between procedure and procept was characterized by Gray and Tall as the “proceptual divide”, a bifurcation of strategy between flexible thinking and procedural thinking which distinguishes more successful students from those less successful (Gray & Tall, 1994). This divergence is evidenced by observable qualitative differences in the strategies employed by the less successful and the more successful students. Various levels of the encapsulation of a procedure can be seen to be successively sophisticated growth of a procept. Skemp (1987) alludes to the notions of procept and proceptual divide when he discusses the difficulties students have in learning to understand mathematical notation and signs. He asks:

“So how can we help children to build up an increasing variety of meanings for the same symbols? How can we prevent them from becoming progressively more insecure in their ability to cope with the increasing number, complexity, and abstractness of the mathematical relations they are expected to learn?” (p. 186).

Although rote-learning of procedures may increase the foundation on which to build, the meaningful learning of procedures is essential for flexible thinking. Some students experience a cognitive shift from concrete actions and processes to abstract cognitive objects able to be manipulated in the mind while others remain locked into procedures. The more successful develop a flexible proceptual system of deriving new knowledge from old and have a built-in feedback loop that creates new mathematical objects. The less successful are caught in a procedural system in which they are faced with harder and harder procedures that eventually result in cognitive overload. Even when the less successful have the procedures available to them, they may lack the flexibility to use them in the most economical and productive way (Gray and Tall, 1994; 1991).

We argue that the proceptual divide—the bifurcation point at which some students function flexibly with mathematical signs as references for procedures and objects and others do not—is a manifestation in a mathematical context of the indexical-symbolic hierarchy. This, of course, no more “explains” the proceptual divide than does naming it as such. What it does, however, is to show that the proceptual divide is one aspect of a more general problem of reference—the difference between indexical and symbolic associations. Secondly, the issue of how brains develop symbolic competence is beginning to be addressed from a neuroscience perspective (Deacon, 1997; Cariani, 2001; for example). There is, therefore, some hope that understanding the pro-
ceptual divide from a semiotic perspective might assist in our understanding of the dynamics of a transition from inflexible procedural interpretations of mathematical signs to flexible conceptual interpretations.

The proceptual divide refers to states of reference. A student who views “–3”, or “2+3”, for example, only as procedures to be carried out, is behaving purely indexically with regard to these marks. The student exhibits only a conditioned response. The mathematical signs, for them, refer only to what it is that one does. This is their reference. A student who sees these signs as procedures, and as objects to be thought about now has symbolic associations for the signs. The signs are embedded in a larger sign system in which reference is not simply of the nature of conditioned response, but is flexible, open-ended, and has contact with other signs, also interpreted symbolically. The essence of the distinction in the proceptual divide is not the communicative skill that is provided by a different interpretation of the signs of mathematics, but in the communicative strategy that a student adopts in interpreting those signs (Deacon, 1997, p. 379).

The symbolic nature of advanced mathematics

Few people would dispute that advanced mathematics is a highly symbolic subject. It’s writings use arcane marks and signs whose meaning is known only to a few cogniscenti. The art resembles a cabalistic ritual practiced by high priests privy to the secret knowledge. We agree that advanced mathematics is highly symbolic, but not in the simplistic sense that it deals in arcane squiggles. Rather, the symbolic nature of mathematics resides in the rich detailed symbolic interpretations of the marks, signs, language of mathematics made by its skilled practitioners.

In advanced mathematical thinking we try to circumvent difficulties in reference by making postulates about things such as “triangles” and distinguishing between axioms and models for those axioms. Dependent on our level or field of discourse we may postulate various entities, such as “lines” and “points” and relations between those entities, such as “incidence”, and then define a triangle in terms of these entities and relationships. This is a highly symbolic activity in the everyday sense: it utilizes the symbols of language in a sophisticated, highly reflective and distilled manner. However, for a reader of such a definition there is no guarantee that the definition is in any way symbolic, and in our experience it is highly likely to be viewed solely as an index—stimulating a conditioned response. Students typically “manipulate” definitions like this to produce “theorems” but their frames of reference are often at a purely indexical level. To speak, therefore, of advanced mathematical thinking as highly symbolic is to take only the point of view of the per-
sons who produce the axioms and definitions, and not those, the students, who attempt to interpret them. This is a clear example of how focusing on the frame of reference for signs overcomes the lack of distinction that results from focusing solely on the physical marks of mathematics. These marks are typically, but unhelpfully, referred to as “mathematical symbols”. We say “unhelpfully” because such a point of view misses the essential distinction in reference between the producer and interpreter of signs—between teacher and student. There is nothing symbolic in association for most people in a text on advanced mathematics, whereas for the writer of the advanced text the whole work has deep symbolic associations. Below are three examples from advanced mathematics of definitions, a theorem, and a proof:

- “A periodic point of period \(n\) of a diffeomorphism will be called dissipative if \(\det(T_p^f) < 1\). Let \(S(f)\) denote the set of periodic sinks of \(f\). That is, if \(p \in S(f)\) and \(f^k(p) = p\) then all eigenvalues of \(T_p f^k\) have norm less than one. “(Guckenheimer, Moser & Newhouse, 1978, p. 91).

- “Suppose \(M\) is a 3-manifold and \(f:B^2 \rightarrow M\) is a map such that for some neighborhood \(A\) of \(\partial B^2\) in \(B^2\), \(f|A\) is an embedding and \(f^{-1}(f(A)) = A\). Then \(f|B^2\) extends to an embedding \(g:B^2 \rightarrow M\). “(Hempel, 1976, p. 39).

- “We have \(a = nd\) so that for \(r \in K[t]\), \(\partial r < n\). Now the field polynomial takes the form \(f_a(t) = \Pi \left(t - r(\theta_i)\right)\) where the \(\theta_i\) run through all zeros of the minimum polynomial \(p\) of \(q\) whose coefficients are in \(Q\). It is easy to see that the coefficients of \(f_a(t)\) are of the form \(h(\theta_1, \ldots, \theta_n)\) where \(h(t_1, \ldots, t_n)\) is a symmetric polynomial in \(Q[t_1, \ldots, t_n]\). By corollary 1.10 the result follows.” (Stewart & Tall, 1979, p. 42).

These three excerpts from the mathematical literature are aimed at graduate students, research mathematicians, and advanced undergraduates respectively. Not only are they replete with mathematical squiggles but they are highly symbolic statements. They are expressed both in language and in mathematical squiggles, all of which intertwine in a system of conventional symbols. To read such statements at anything beyond an indexical level requires a sophisticated apparatus of symbolic reference in mathematics.

A student is literally incapable of forming the appropriate symbolic links—those intended by the writers of these statements—if their interpretative response for even a few of the words or squiggles is still largely indexical. A massive leap is required to uncover the higher-order associations in the rich symbolic webs required to read these statements symbolically. That is partly why mathematics is hard for most people: at this level: it requires continued and practiced nimbleness.
in symbolic interpretation. Students of advanced mathematics need, in order to function fluently with understanding, a dedication to continually expanding their symbolic competence.

Thus, an exclusive focus on pictures, likenesses, icons, or an exclusive reliance on learned associations, will not be adequate to function flexibly at this level of mathematical thought. This is not to say that iconic pictures and reflex associations don’t help—they do. But the nature of mathematics is inherently symbolic. Not in the trivial sense that written mathematics uses lots of squiggles, but in the deeper semiotic sense that a student needs a high level of symbolic interpretation to begin to understand advanced mathematical statements. A student’s focus of attention needs to shift, from direct, indexical, references to higher-order associations between those direct references. The mathematical reality is elsewhere than in the embodied objects, which are merely a skeleton on which deeper symbolic insights are supported.

**Beyond symbols**

The question we address in this section is whether interpretations of mathematical marks and diagrams offers the possibility of a semiotic level that goes beyond symbolic thought as it is understood in a language context. The symbolic connections of language are conventional, those of mathematics only partly so. At a simple level the signs of algebra sometimes state necessary, and not simply conventional relationships. For example, to say that for any real numbers $x$ and $y$, the expression $(x + y)^2$ is equal to the expression $x^2 + 2xy + y^2$ is not a conventional relationship. It depends on other conventional relationships between symbol tokens, such as when, conventionally, we take two such expressions to be “equal”. Beyond that, the statement that these two expressions are equal is a result of a deduction, not of convention. In this, mathematics differs fundamentally from natural language. The definition of a prime number as a positive integer with exactly two factors is conventional, but the statement that 7 is a prime number is not. Neither is the statement that there are infinitely many prime numbers a convention, though many students probably see it as such. Nor is the deeper fact that the number of prime numbers less than $n$ is asymptotically $n/\log(n)$. Mathematics deals also with necessary relationships between things conceived of at a symbolic level. Mathematics therefore has the possibility of a supra-symbolic interpretation, dealing with a level of semiotic interpretation that organizes symbols themselves.

The transition from a largely indexical way of thinking about signs to symbolic forms of reference is a sophisticated process. That so many human beings manage this transition as they learn to speak tells us something about the profound abilities of human brains. The shift from
indexical to symbolic reference requires a shift of the focus of attention. The transition requires us to not to be too focused on or ensnared by the sharp tip of an icon or index’s arrow of reference. We need to shift focus to higher-order associations among the signs themselves, not so much to learn but to learn to pay attention.

We hypothesize that a similar shift in attention is required for students of mathematics to focus on the necessary connections that exist between certain mathematical symbols. The transition is from viewing all mathematical statements as conventional, as most are in everyday language, to a focus on the symbolically mediated yet necessary relationships. In our experience teaching undergraduates this is harder than a shift from indexical to symbolic thinking in mathematics. Discovering the necessary as distinct from conventional relationships between mathematical symbols seems to be more difficult for students to grasp, and much rarer in its manifestation than an ability to understand mathematics at a symbolic level.

Do our brains have the apparatus to do this? Some people’s do, otherwise mathematics would have no theorems. What then, in our brains, allows us to do proof, to discover the necessary logical relationships between conventional mathematical symbols, symbols interpreted in the rich connected sense of semiotics?

References


