Theoretical Framework & Purpose

Thompson and Saldanha (2003) argued that the instruction of fractions as quantities needs to be considered in multiplicative ways. In their view, too many U.S. children are taught about fractions and fraction operations in ways that do not support the development of more complex mathematical ideas – in fact, some of the approaches hinder further conceptual development. For example, conceptualizing multiplication as repeated addition is problematic both because it confuses operation (repeated addition) with conception (making groups of a quantity and replicating those groups) and because multiplication conceived of as repeated addition is an insufficient model for interpreting a quantity such as 3/4 * 2/7. In response to these shortcomings, Thompson and Saldanha suggest that school mathematics should provide students with opportunities to learn about fractions in ways that highlight the multiplicative relationships of quantities. By doing so, they argue, many students’ misconceptions could be avoided.

However, the instruction necessary to achieve this vision rarely happens in U.S. classrooms (Thompson & Saldanha, 2003). The literature on teachers and their classroom practices shows that too often teachers’ mathematics is not conceptually grounded. For example, Ma (1999) found that 43% of the American teachers in her study correctly solved a problem such as $1 \frac{3}{4} \div \frac{1}{2}$ and only one of the 23 American teachers in her study was able to generate a word problem situation for fraction division. Similarly, Post, Harel, Behr, and Lesh (1991) found that of the 218 fourth through sixth grade teachers to whom they administered an assessment on ratio and rate, 28% were unable to respond to a question such as "Melissa bought .46 pounds of flour for $.83. How many pounds of flour could she buy for one dollar?" and only 45% answered the question correctly. Only 10% of the teachers provided a mathematically sound explanation for their solutions.

This lack of conceptual understanding is alarming given that researchers are beginning to demonstrate the importance of teachers’ mathematical knowledge for teaching (MKT). This literature suggests that teachers need to have not only a knowledge...
of mathematics that all adults need, but also need additional, specialized knowledge of mathematics that allows them to teach concepts to students (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). Research focused on MKT has begun to form a clear relationship between teacher knowledge and student learning. For example, Hill and her colleagues (Hill, Rowan, & Ball, 2005) found that teachers’ MKT as measured by the elementary grade LMT instrument for measuring MKT had the same effect as socio-economic status on third-grade students’ mathematics performance. In a different line of research, Charalambous (2010) considered the impact of MKT on teachers’ practices. He found that teachers with higher MKT selected and enacted tasks with higher levels of cognitive demand than teachers with lower MKT, thereby increasing learning opportunities for students.

To ground fraction instruction in multiplicative reasoning, shortcomings in the teachers’ knowledge must be addressed. A small, but growing, body of research indicates that teachers who have opportunities to develop their own conceptual knowledge in professional development are likely to change their practice in light of their learning (see Mewborn, 2003 for overview of such studies). While we know that teachers struggle to think about fractions as quantities and to understand operations with fractions multiplicatively, little is known about how teachers reason with fractions and where their misconceptions and shortcomings might be.

In this study, we report an exploratory analysis of teachers’ reasoning by considering their discussions in a professional development setting designed to promote conceptual understanding of fraction operations. Our analysis builds from Thompson and Saldanha’s (2003) assertion that understanding requires a “web of meanings” between ideas such as measurement, multiplication, division, and fractions. We also build from Lamon’s (1994) assertion that connecting the ideas that are prerequisite for fractions is key for understanding them. We consider some of these foundational understandings found in the literature including the role of the referent unit in teachers’ talk about fraction division, the role of representations such as the double number line in supporting teachers’ understanding, the purposeful use of proportional reasoning in fraction division, and the use of contextualizing as a tool for reasoning about fraction division.

Methods & Data Sources

This study was part of the larger Does it Work project which focused on understanding what teachers learn in professional development, how that learning connects to their classrooms and whether there is any effect of the teachers’ learning on student performance on assessments. The current study considers a single professional development course, InterMath – Rational Number (http://intermath.coe.uga.edu), which focused on technology-enhanced investigations for developing teacher conceptual knowledge of fractions and proportions. The course was offered in a large urban school district in the southeast. Teachers met in a computer lab at the district offices three hours per week for the 14 weeks of the course. Thirteen of the participants were mathematics teachers in grades five through seven and one worked in the district offices. The participants’ experience levels varied from novice (less than three years) to highly experienced (greater than 11 years). The facilitator of the course was a former classroom teacher and member of the larger research team. The explicit goals for the course were to
further develop participants’ understanding of referent units, develop fluency with multiple drawn representations of fraction operations, and refine proportional understandings in the context of middle grades mathematics.

The primary data sources for this analysis were videotapes of the class meetings. Each class was videotaped by two cameras, one focused on participants and their work and the other on the facilitator. The videos were combined to create a reconstructed view (Hall, 2000) using picture-in-picture technology. In addition to the video, partial transcriptions of the classes were created, broken into thematic episodes, and time-stamped for easy retrieval and review of the original footage. In addition to these videos, the larger research project also conducted weekly phone interviews with each participant, assessments of teacher understanding of the concepts included in the course, and cognitive interviews with a subset of the teachers that allowed the teachers to explain their reasoning on the assessments. All interviews were fully transcribed. These secondary data sources were used to help identify potential areas of interest in the set of videotapes.

We chose the four class sessions (weeks 5, 6, 8 and 9) that directly addressed division of fractions. Each of the researchers reviewed one video individually, noting episodes of rich dialogue and making a list of initial themes present in the data. At later meetings, these emerging themes were discussed and modified to find categories of participant meaning that cut across all three classes (Coffey & Atkinson, 1996). The emerging themes included a view of fractions as numbers versus quantities; the role of proportion as a tool for solving tasks; and conceptual understanding of fraction division (or lack thereof). Together, the three researchers reviewed and discussed several relevant episodes in each lesson that had been identified during the first pass through the data. We noted the trends individually and as a research team using memoing techniques (Strauss & Corbin, 1998). A subset of the themes we found forms the analytic structure that this paper will discuss.

Results

Our analysis focused on five fraction division tasks that allowed participants opportunities to use a variety of approaches and representations. In all five tasks, the teachers were challenged to make connections to other mathematics concepts including proportion and multiplication. Below are some of our initial findings about how teachers’ understand of fraction division. We conclude with a brief discussion of the implications of this work.

Referent Unit

As Lamon (1994) noted, fraction division inherently requires renorming because the divisor becomes the referent unit for the dividend. That is, in a problem such as 2/3 ÷ 2/5, the quotient does not refer to the original whole, rather it can be thought of as the number of 2/5 that are included in 2/3 of the whole. This is a point few teachers’ discuss. Because referent units were one of the three overarching themes of the professional development course, the participants had explicit discussions about them in InterMath – Rational Numbers. And, these conversations uncovered that the participants generally struggled with identifying the referent unit of the quotient.
A task presented during the fifth week of the InterMath course provided an opportunity for a rich discussion of referent unit. During the Exploring Division task, participants examined the similarities, differences, and patterns in the following set of division problems: $2 \div 3$, $2 \div 1/4$, $2 \div 3/4$, $2/3 \div 3/4$, $1/3 \div 3$, $2/3 \div 3$, $2/1 \div 1/4$ and $2/3 \div 4$. When the whole class came together to discuss $2/3 \div 3/4$, the participants’ varying understandings of referent unit for the quotient came to light. Keith, Donna, and Claire provided models of this problem in which they clearly identified the quotient ($8/9$) as referring to $3/4$. However, during these explanations several of the other participants were unable to conceive of this idea. For example, Carrie expressed her frustration with the problem saying she was “so lost it’s unreal” as Keith explained his model to the class.

The participants’ exploration of referent units continued. In week 8, the facilitator posed the following problem: What is the referent unit for each number in $2/3 \div 1/4 = 8/3$? After being presented with the task, one participant stated that she “was wishing for a good, working definition of referent unit”. Several of the participants spoke up in agreement, acknowledging they too would like a definition of referent unit. In the discussion that followed, various different definitions of referent unit were offered. For example, Donna explained that each number in the problem had a referent unit of one, which she elaborated to mean “the number one”. King emphasized that the referent unit for each was one, but added that he was not sure whether that meant the number one or one whole. Keith identified referent units incorrectly and then tried to reason about what the referent units might be by thinking through the multiplication problem $2/3 * 4 = 8/3$. From these conversations and others like them, it was clear that keeping track of the whole was a deliberate and complicated effort for these participants. This was true even after several weeks of thinking about referent units and whether the participant had the ability to solve the items in conceptually-grounded ways or not (as Keith did with his model in week 5).

By the ninth class, we saw evidence that the teachers could reason about the referent unit in division situations because they were able to use drawn representations to identify why $7 \div 3/4$ has an answer that includes $1/3$. For many, this was new reasoning as a result of the professional development.

**Representation**

Throughout the course, the participants were encouraged to use to a number of drawn representations for thinking about relationships between quantities. For fraction division, the instructor highlighted double number lines (DNL) and area models. From our perspective as the course designers, the DNL offered an opportunity for the participants to consider division as a kind of proportional reasoning. We believed this was especially true for partitive division situations.

In week 8, the discussion of a problem asking how many cans one container holds if 6 containers fill two and a half cans led to Mike declaring “the double number line works for proportions but not for division.” Sharlene also commented that she did not know how to use the DNL, saying she was “stuck at this point to find 1 container.” She discovered through trial and error that dividing by seven allowed her to see the

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2 All names are pseudonyms.
relationship between containers and gallons, but asked rhetorically, “Why does 7 work? I don’t know.”

In addition to the use of DNLs, participants frequently used area models to represent fraction division. Though most of the participants seemed more comfortable using area models to represent quotative division situations, some of the participants were able to use area models for partitive situations as well. For example, when the facilitator posed the following partitive division problem: If 1/4 of a serving is 2/3 of a brownie, how many brownies make a whole serving? Mike explained his model shown in Figure 1, saying, “that whole thing (the large rectangle) is one serving and so each one of those things (columns) is 2/3 of the brownie…so 2/3 + 2/3 + 2/3 +2/3 = one serving”. Donna, and Claire provided a slightly different model (Figure 2). Donna explained that she and Claire, “drew a brownie and divided it into thirds…that’s (the 2/3) 1/4 of a serving, so I need four of those for a whole [serving]”. Although both of these models were referring to a partitive division problem, the participants were reluctant to call this division, saying that they did not see the division. Instead, they approached it as a repeated addition or multiplication task in which they iterated the quantities to find their answer.

![Figure 1](image1.png)

*Figure 1. Mike’s area model representation of the brownie problem*

![Figure 2](image2.png)

*Figure 2. Donna and Claire’s area model representation of the brownie problem.*

**Proportion as a Tool**

In looking at a partitive representation for 3 ÷ ½ (Figure 3), the teachers were given a double number line divided into six equal lengths with a label at the halfway point that said ½ on the top line and 3 on the bottom line. Then, at the end of the double number line, there was a 1 on top and a 6 on the bottom. Donna and Claire discussed this model at length. Donna was having a difficult time understanding how the model represented the problem. She continued to point out that she could understand the model if she could interpret the vertical marks as 1/2s, ignoring the given quantities, then the model shows 6 halves are in 3. Claire tried to explain the model to Donna by stating that it represented the question, “if a half of something equals three then the whole of it equals what?” However, this interpretation did not make sense for Donna in the context of the
mathematical statement $3 \div \frac{1}{2}$. Claire continued to help Donna by writing in the missing quantities on the DNL, which allowed Donna to understand the proportional relationship modeled on the double number line. However, Donna still could not relate the model to division, stating, “It is not a method I would choose…I can see it, but it doesn’t go very far towards helping me understand [the problem].”

![Double number Line representation of $3 \div \frac{1}{2}$](image)

*Figure 3. Double number Line representation of $3 \div \frac{1}{2}$*

Several of the participants demonstrated an intuitive sense of the relationships between quantities in the division tasks when they were provided in context, however, these intuitions were not robust enough to support solutions. That is, the participants seemed to understand that the cans and containers could form a ratio and that cross-multiplication could be used to find the result. However, the teachers did not understand how this related to division nor were they able to apply this approach in ways that led to the correct answer. In weeks 6 and 9, the teachers worked on similar tasks looking at the relationship of one quantity with respect to another. For example, in exploring how many cans one container would hold given that 6 cans would hold 2 ½ containers, Sharlene set up the proportion $\frac{6}{2} = \frac{?}{1}$ to solve the problem. However, when Sharlene used her calculator, she divided 2½ by 6. Although her written work was evidence of an intuition to approach the problem proportionally, her execution suggests a misunderstanding of the division situation. Further, she seemed to lack the understanding that there is a reciprocal relationship between the ratio of cans to containers and the ratio of containers to cans.

**Context**

We found that teachers’ reasoning was constrained by the context of a task when they attended to the context and tried to make sense of the mathematics entailed by the task within this context. For example, sometimes attention to a particular context seemed to limit teachers’ ability to use both the area model and DNL representations of division. In one such instance, Donna created a context as part of her explanation of partitive division; however, that context prevented her from being comfortable with thinking of the task as a division item. Donna and several others involved in the discussion expressed frustration with trying to think of the task as division because to them, the context of the problem meant that it was a proportion and therefore was not about division.

On the other hand, when teachers ignored the context and relied on algorithms or symbolic manipulation to solve the tasks, they sometimes made erroneous claims that were illogical within the context. In these cases, it often appeared that teaches did not pay explicit attention to the task context and as a result failed to recognize these errors. For example, in Sharlene’s approach to the cans and containers task, described in the Proportion as Tool section, it was clear that she stripped away the meanings of the
numbers as she created the proportion. Had she retained the context of cans and containers, she may have been more able to see that she was not calculating the answer she intended.

Attending to the task context seems to simultaneously support teachers’ sense making and constrain their flexibility in thinking about mathematical ideas such as division. This may be because some contexts are more likely to be interpreted as partitive (or equivalently, quotative) division, and as a result some teachers may not find both the area model and the DNL equally supportive of their thinking about the contextualized division problem.

**Significance & Implications**

This work is significant because it begins to address the gap in the literature about how teachers understand this problematic content and provides insight that can drive professional development.

We uncovered four big ideas that proved to be challenging for our teachers. First, this work suggests that there are some key elements of fraction division that are worthy of serious consideration in professional development contexts. Referent unit is one key building block that is necessary for making sense of fraction division. Further, the use of representations as tools for sense making and as tools to support communication seems warranted. Because of our participants’ use of representations, we were often able to understand how they were thinking about tasks. They also were able to ask each other mathematical questions about ideas and provide explanations that may not have otherwise been given. Third, supporting teachers in developing a meaningful understanding of partitive division includes helping them connect ideas about division and proportional relationships. For our teachers, this was new knowledge. Finally, careful consideration and use of contextualized problems seems important for grounding teachers’ reasoning in the meaningful mathematics. The use of context in professional development seems to present a special challenge in that context often provides a way for teachers to develop meaningful solutions but some task contexts are not sufficiently general to support the meaningful use of all the relevant mathematical models. Professional developers must be aware of these challenges as they select tasks that will support teachers in developing connected understandings.

In terms of understanding how teachers make sense of the content, our findings are consistent with the literature on students’ proportional reasoning and fractions. In short, we found that our teachers struggled to connect the ideas of fraction division and proportional reasoning. These missing connections impede teachers’ ability to help students develop webs of meanings about mathematics (Thompson & Saldanha, 2003). Our participants struggled to expand their own webs of meaning to incorporate the relationship between proportional reasoning and fraction division, evidence that points to a clear need for more professional development of this kind. That is, professional development that engages teachers in exploring mathematical concepts in multiple ways and sharing their understandings about their approaches with each other.
Works Cited


