Preliminary Observations on Teachers’ Multiplicative Reasoning: Insights From the Does it Work and Diagnosing Teachers’ Multiplicative Reasoning Projects

Technical Report #6 | April 2010

Chandra Hawley Orrill
University of Massachusetts Dartmouth

Andrew Izsák, Allan Cohen & Jonathan Templin
University of Georgia

Joanne Lobato
San Diego State University

GRADUATE & POSTDOCTORAL ASSOCIATES
Zandara de Araujo, Laine Bradshaw, Rachael Brown, Günhan Çaglayan, Bridget Druken, Erik Jacobson, Soo Jin Lee, Susan Sexton Stanton, Aijun Wang

“Democratizing Access for All Students to Powerful Mathematical Ideas”
Preliminary Observations on Teachers’ Multiplicative Reasoning

Lead Team Members

Chandra Hawley Orrill
STEM Education Department & Kaput Center for Research and Innovation in STEM Research, University of Massachusetts Dartmouth

Andrew Izsák, Allan Cohen & Jonathan Templin
Mathematics and Science Education Department, Georgia Center for Assessment University of Georgia, & Educational Psychology and Instructional Technology Department

Joanne Lobato
Center for Research in Mathematics and Science Education
San Diego State University

Acknowledgments:
This material is based upon work supported by the National Science Foundation under Grant No. REC-0633975, DRL-0822064, & DRL-1036083. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Purpose & Disclaimer
This technical report is being compiled based on preliminary work of the project personnel for the purposes of highlighting some of the key understandings uncovered through our initial analysis activities related to the two NSF-funded projects. This report is provided solely as information to support the design of professional development programs. This document is being compiled by Chandra Orrill on behalf of the project team members and she assumes all responsibility for any errors or omissions. As this is a preliminary document, we request that it not be copied or distributed except as expressly approved by members of the Lead Team.
BACKGROUND

This document summarizes some key findings from the Does it Work and Diagnosing Teachers’ Multiplicative Reasoning projects. These projects, both funded by NSF, aimed to explore how teachers understand and come to know ideas central to rational numbers.

Does it Work Project Overview

Does it Work: Building Methods for Understand the Effects of Professional Development is a professional development research project in which we consider three key questions: (1) what do teachers learn in professional development; (2) how does their learning translate into their classroom practices; and (3) assuming there is an impact on practice, what are the effects on students’ achievement. To this end, we studied professional development experiences designed and offered as part of this study. The experience, InterMath – Rational Numbers, was designed for middle school teachers and offered four times over a two-year period. Our analysis focuses on the two classes offered in the second round of implementation in which the course focused on multiplication and division of fractions and decimals, and proportional reasoning. The InterMath – Rational Numbers courses are outlined on the InterMath website (http://intermath.coe.uga.edu/workshop/homepg.htm). The three stated goals for the professional development were to support teachers in improving referent unit reasoning, broadening their understanding and use of drawn representations, and proportional reasoning. The courses met three hours per week for 14 weeks. A variety of qualitative and quantitative data were collected during these classes and analysis is currently ongoing for all three research questions.

Diagnosing Teachers’ Multiplicative Reasoning Project Overview

The Diagnosing Teachers’ Multiplicative Reasoning project is focused on the development of an instrument designed to measure teachers’ multiplicative reasoning that capitalizes on Cognitive Diagnostic Models. This approach requires critical components (also called attributes) for the domain of multiplicative reasoning in order to generate an organization of the domain that will allow the statistical model to identify teachers who have mastered or not mastered those aspects. In order to map the domain, we have engaged in a two-pronged approach of consulting the research literature on teachers’ and students’ knowledge and engaging in clinical and cognitive interviews with teachers around particular subtopics. At this point, we are identifying not only those attributes of reasoning related to fractions and proportions that seem to help sort teachers, but also those that seem to be measurable in a paper-and-pencil assessment. Our primary data sources for this study include cognitive interviews around a series of draft items initially developed based on the literature and a set of clinical interviews that were conducted to better understand teachers’ approaches to fraction and proportion tasks.

PRELIMINARY FINDINGS RELATED TO CONTENT

Referent Units and Norming

Many of the teachers in our interviews and InterMath courses struggled with referent units and renorming. This was evident in problems such as the Beaker Problem (see below) and became increasingly problematic as the teachers tried to multiply and divide fractions and decimals. While the literature treats renorming and referent unit separately, we found the two problems were closely related as the teachers moved through a variety of questions.

Interpretations of Fractions

Many of the teachers in our interviews and in our InterMath courses relied primarily on only one conception of fractions, the \( n \)-out-of-\( m \) conception. We found that both in the ways they thought about fractions and operations with them, the teachers spoke of fractions as a number of pieces out of a number of pieces. They seemed not to conceptualize fractions with a multiplicative interpretation in which a value such as \( \frac{3}{4} \) is conceived of as 3 one-fourths, where each one fourth is in a multiplicative relation with the whole. This second conceptualization is referred to as a \( \frac{1}{b} \)ths of \( m \). While we have noted that many teachers do understand both conceptions of fractions, their over-reliance on the \( n \)-out-of-\( m \) conception has implica-
tions operating with and on fractions and for moving into other aspects of multiplicative reasoning.

**BEAKER COMPARISON**

Sam and Morgan are comparing the amount of liquid in their beakers as shown in the diagram below. Sam claims that Morgan has 20% less than she has. Morgan claims that Sam has 25% more than she has. Who is right?

![Beaker Comparison Diagram]

Figure 1: The Beaker Problem is an example of a task that requires renorming.

**Meanings for Division**

Many teachers we have interviewed do not have a conceptual basis for fraction division. If teachers do have a conceptual basis, they tend to rely on measurement (quotative) models for division. Thus, they view division questions as asking ‘how many $x$ in $y$?’ The teachers, generally, do not have a partitive interpretation of division and struggled with this interpretation in class. The partitive-interpretation asks “if $x$ units correspond to $y$ units, how many units correspond to 1 unit?” The teachers similarly had little understanding of how to model either model of division using drawn representations, and many were unable to interpret the meaning of fractional quotients (e.g., in $1/3 \div 2/5$, why is the quotient $5/6$ – a number that is neither thirds nor fifths.) This is an important example of limitations in teachers’ understanding of referent units.

**Ratios as Composed Units that Can Be Iterated and Partitioned**

We have noticed that teachers often do not reason with ratios as composed units. Instead, they either reason about each quantity separately, using the rule of ‘what you do to one number you do to the other’ or other rote reasoning. Further, teachers seem to be limited in their ability to conceptualize partitions other than halves or iterations that are not whole numbers. For example, many of the teachers struggled to iterate or partition a fractional amount (e.g., making $\frac{1}{4}$ of a batch of a recipe or making 2.5 batches of a recipe).

**Composed unit** reasoning requires that the teacher has coordinated the two values in the ratio into a new unit (for example, oil and vinegar may coordinate into a new unit called “batch”). The invariance of that composed unit ratio can be maintained through iterating or partitioning the composed unit.

**Scaled Factor**

Many of the teachers we interviewed were able to determine ways to use scale factors to approach a number (e.g., given a 1:8 ratio, how would you calculate the amount needed for x:50?) However, they struggled with determining the necessary fractional amount directly—instead opting to get close by dividing 50 by 8 to see how many whole number iterates would be needed, then figuring out the remaining portion separately.

**Within and Between Measure Spaces**

Teachers generally seem to understand that comparisons can be made within measure spaces or across those spaces (e.g., vinegar:vinegar and oil:oil or vinegar:oil and vinegar:oil). However, they tend to have a strong preference for one of these relationships over the other—depending on the teacher and/or the problem situation. Further, they seem able to apply scale factor reasoning to both so that they see how many times bigger x is than y in $x/y$ or how many times bigger x is than a in $x/y = a/b$.

If one thinks of proportional relationships as being multiplicative comparisons of two quantities (that is, comparisons that ask how many times larger is one than the other), one can make those comparisons within space, which is comparing the same units (e.g., original amount of vinegar to new amount), or they can be across space in which the units are different (e.g., relationship of vinegar to oil).
Reciprocal Relationships of Relative Size
Like students, many of our teachers demonstrated no understanding of this notion. When confronted with situations that suggest the use of this knowledge, teachers rely on calculation – and seem to not tie it back to the reciprocal relationships of relative size.

Related to this, some of the teachers we interviewed could not see the value in determining unit rates. They stated that while they could calculate a unit rate, they did not know why they would want to know the unit rate to help them solve particular problems.

Algorithmic Formats

Struggles to form \( y=mx \) algorithm
In our interviews with 15 teachers and in observations of two InterMath courses, we noted that teachers who have not explicitly taught the concept, struggle to formulate equations that describe proportional relationships. In fact, we saw one teacher begin her equation by writing \( y=x+2 \) before she even started thinking about what \( m \) could be. This indicated that she lacked a robust understanding of what it means for a relationship to be proportional.

Similarly, we noted that when given a relationship in \( \frac{a}{b} = \frac{c}{d} \) form, some teachers struggled to write a valid equation while others could not explain what was wrong with an incorrect equation. For example, in a comparison such as \( \frac{3}{2} \) cinnamon to \( \frac{7}{2} \) orange, the teachers could not explain why \( \frac{3}{2} \) \( C = \frac{7}{2} \) \( O \) was incorrect – and some teachers even accepted it as correct because they felt it showed a valid comparison. For those confused teachers who were asked to revise the equation, suggestions for improvement included using only one variable (e.g., \( \frac{3}{2} x = \frac{7}{2} x \)).

Knowing \( \frac{a}{b} = \frac{c}{d} \)
As mentioned previously, the teachers generally know how to set up proportional expressions. Teachers also have reasonable explanations for how to set up the valid equivalent ratios.

Meanings for Equality
In general our teachers struggled to verbalize what equality means in the case of equivalent ratios. When prompted, many were able to identify that the taste stays the same or the miles per gallon remain unchanged, but they lacked any verbalized ability to express what it means for the ratios to be equal. Some meanings that teachers did offer were equal sign as a comparison point (how does the value on one side relate to the value on the other side), equal sign as indicating equality (treating the ratios as equivalent fractions), or equal sign indicating containment (one ratio is contained in whole number iterates of the other).

Reinterpreting Ratio as a Fraction
Most of the teachers we interviewed were unable to reason through what the ratio was in terms of a fraction of some whole in the problem situations. For example, if the ratio 2 units of vinegar to 5 units of oil was given, and the teachers were asked what there was 2/5 of, the replies were confused. Most teachers asserted that 2/5 was an inappropriate quantity to think of in fractional terms because it was a part:part comparison and fractions are a part:whole comparison. Further, they stressed that the appropriate comparison was 2/7 and 5/7 since the 2 and 5 combined to make 7 parts. We believe this confusion is related to the inability to find or reason with unit rates mentioned previously.

Differentiation Between Ratios and Fractions
There are a host of interesting issues related to teachers’ conceptions of fractions and ratios and the relationships between them. In general, teachers seem to have not explored this relationship in meaningful ways, but, when faced with questions about the relationship, they find the conversation interesting. The “Selecting appropriate representations” section discusses some aspect of this in more detail. In general, teachers do not have a way to differentiate the mathematical activity involved in “tripling a batch” from multiplying by 1 to find an equivalent fraction. Therefore, they are using equivalent fractions
as a method for thinking about what happens mathematically when we triple something. This is likely somewhat confusing for students – after all, the point of multiplying by 1 is to further partition the same whole to rename a given part whereas tripling or doubling means actually making more of something than there originally was. Clearly, this becomes an issue related to referent units, which are also problematic for teachers.

Proportionality as Linear Relationship
In general, the teachers to whom we talked expected that a proportion was a linear relationship. However, most seemed to lack specific knowledge about particular aspects of the line that are critical (such as going through the origin and recognizing the constant of proportionality as the slope). It is unclear from our interviews whether teachers would understand how to determine the slope from the $y=mx$ equation. We also noted that teachers do not feel it necessary to have an $x$ and a $y$ in proportional equations. (Teachers also have relatively rote understandings of slope that do not support reasoning about the covariation of quantities in meaningful ways. Instead, they tend to think rise-over-run in an algorithmic way that leads us to question whether they would be able to reason about values of points that happen to fall between whole-number iterates of the slope.)

Appropriateness
In our interviews, we found that a number of teachers struggled to identify appropriate and inappropriate situations for direct proportion. For example, in word problems, if 3 values were given and one was not, they often identified it as appropriate even when the situation described was an inverse proportion. Similarly, in items that used other formats, such as an image showing one pile of blocks as compared to a pile of different blocks, the teachers did not invoke proportional reasoning because they did not readily identify the situation as being a proportional one. Instead they simply counted blocks.

General Thoughts on Proportion
Generally, cutting across these categories, it seems that the teachers lack a coherent definition of proportion. They understand different parts and pieces of what it means to be proportional, but have trouble coordinating those across a variety of problem types. Some teachers know how to create $y=mx$ equations while others are constrained to $a/b = x/y$. Some teachers understand that proportions result in linear graphs, but they do not seem to understand that those graphs must cross through the origin – and, likely, have not thought about why a direct proportion would always pass through the origin. Some teachers seem to understand slope, but do not connect it to the $m$ in $y=mx$. Many seem to lack a sense of covariation as they think of the quantities as varying only within measure space.

In our analyses to date, which are preliminary, we have noticed that most teachers rely on cross multiplication and, maybe, one other approach to solving proportion tasks. We are learning that they know about more solution paths than these, but only use those other approaches when compelled to or when asked to analyze students’ work.

PRELIMINARY FINDINGS RELATED TO CONTENT

Selecting Appropriate Representations
In our cognitive interviews with teachers about proportional situations, it has become apparent that teachers, in general, have not had much opportunity to critically reflect on the trade-offs of different representations for teaching particular topics. This means that they may be ill-equipped to make high-quality instructional decisions about them. As a concrete example, we provide two diagrams representing the idea that $2/5 = 6/15$ in terms of ingredients in a recipe. One of the diagrams uses a bar model to show a quantity of 2 sections out of 5 shaded in a rectangle and a second rectangle that has the same
quantity shaded, but with each of the 5ths further partitioned so that the shaded amount now represents 6/15. The second diagram uses small rectangles to show 2 of one item to 5 of a different item all in an oval. This oval is then repeated 2 times to show the iteration of the quantity.

In our interviews with 15 teachers, we noted that the teachers were generally able to make sense of what the representations showed, but clearly had never thought about the instructional implications of one diagram versus the other. Further, the teachers were generally unable to identify the trade-offs of using one diagram versus the other. (Note that this also relates to teachers’ lack of clarity on the distinctions between fractions and ratios.)

While we only have incidental data to reinforce this, we suspect that this general lack of careful analysis of affordances and limitations of representations for instructional purposes has been largely overlooked by teacher preparation and professional development efforts. This, like other aspects of the domain of multiplicative reasoning, seems to reinforce the notion that teachers often have a lot of pieces of understanding about mathematics, but they are not well connected in ways that would support them in making instructional decisions and supporting student learning.

Translating and Transforming
In our initial analyses of the InterMath classes and the clinical interviews conducted as part of our instrument validation efforts, we were struck by teachers’ inabilities to move within and between different representations. We found that many teachers’ abilities to interpret and reason with representations were very tied to particular situations. For example, nearly all of our teachers were generally able to make sense of area models for multiplication of fractions. However, almost none of the teachers could make sense of area models for decimals—in part because there was a shift from the area presented representing one whole with the factors showing parts of part of that whole to a representation in which the whole was absent and only the parts-of-parts were represented. Similarly, while teachers were generally able to reason with area models for multiplication, they were unable to make sense of them for division. This related to one of two key factors: either the teachers were unable to renom in order to interpret the shaded regions in appropriate ways or they tried to apply the process of multiplication to the model in an effort to reason about division.

In the sample of teachers analyzed for the representation studies, we did notice that teachers with strong mathematical understandings were able to apply their understandings to novel representations in reasonable ways most of the time. This suggests that the issues with representations are tightly tied to broader issues of mathematical understanding.

Reasoning with Representations
In our analysis of teachers’ reasoning about a subset of multiple choice assessment items, we noted that teachers often use inefficient or inappropriate approaches to solving problems in which there are representations. The problem with these approaches is that they all work sometimes, which means that the teachers are not placed in a position that challenges their understandings of their approaches. For example, in our interviews around area and linear models for fraction and decimal operations, we noted that teachers failed to use reasoning grounded in attending to referent units in 66% of the instances we analyzed (58 of 88 instances). In 40 of these 58 instances, one of four strategies was used:

- Identifying requisite features—the teacher might look at an area model for fraction multiplication to determine whether they could see each factor as well as the solution highlighted in that diagram. This led to teachers selecting diagrams that were divided in inappropriate ways.
- Looking for a diagram that matches a solution—the teachers selected diagrams that had the cor-
rect amounts shaded in the diagram without regard to the process demonstrated in the image. For example, if a particular length was highlighted on a number line, the teachers did not seem to attend to whether it was a product or a difference in the ways that number line was labeled.

- Using the process to try to select a correct diagram—the teachers chose diagrams in which they could see the process of solving the problem rather than those that actually modeled the process. For example, one of the more effective uses of this approach was in partitive division situations in which teachers could see the process of multiplying by the reciprocal in the ways the number lines were labeled.

- Measuring—teachers created informal measuring devices by marking their papers or using their fingers to compare highlighted areas. However, they failed to account for the lack of precision in their uses of these tools, thereby rendering them problematic for determining answers.

In contrast, the teachers who did use referent unit reasoning either demonstrated inflexibility with them (e.g., they failed to renorm appropriately) or they were able to reason logically and correctly about the situations. Inflexibility was present in 8 of 30 instances of referent unit reasoning.

Double Number Lines
As with other representations, the Double Number Line is only minimally discussed in the literature. In our InterMath courses offered as part of Does it Work, we noted that teachers were unable to readily make sense of the double number line. Consistent with our assertions above about proportional reasoning, we noted that teachers are not sure how to interpret this representation. Our assertion is that teachers’ general lack of a robust understanding of proportional relationships allows them to apply only partial understandings to most tasks and that the understandings they rely on for double number lines. In general, the issues the teachers had were related to a failure to apply composed unit reasoning to the number line, a failure to reason about partitioned values on the number line, and a lack of attention to the details that were important in coordinating the covarying quantities along the number line. In fact, without focused support in one of our InterMath courses, the teachers were unable to use the double number line as anything other than a picture of the result teachers calculated using cross-multiplication.
Sponsors: National Science Foundation, Grant No. REC-0633975, DRL-0822064, & DRL-1036083

Prime Grantees: University of Massachusetts Dartmouth, MA, USA
San Diego State University, San Diego CA, USA
University of Georgia, Athens, GA, USA