The area of a lune

The area of a sphere of radius $R$ is $4\pi R^2$.

Divide a hemisphere into $q$ equal lunes by drawing great circles all from one point on the great circle which forms the boundary of the hemisphere. The lunar angle of each is $\pi/q$.

Area of each lune is $2\pi R^2/q$.

Now if we look at the union of $p$ of these lunes, we get one lune with lunar angle $p\pi/q$, and the area will be $2p\pi R^2/q$.

Thus if $\alpha = p\pi/q$ is the lunar angle, then the area of the lune is $2R^2\alpha$. Thus

area(lune) $= 2R^2$ (lunar angle).

A nice proof the area of spherical triangles can be found at:
http://planetmath.org/encyclopedia/AreaOfASphericalTriangle.html

See handout for answers to your explorations on the Lenart Sphere.
Week 4: Explorations in Projective Geometry

Projective Geometry

Projecting a Sphere onto the Plane and Geometric Inversion

Projection:

1. Imagine the stereographic projection of a sphere onto a plane, which projects any point on the sphere onto a plane by using the point where the line joining that point to a fixed point (called the North Pole) meets the plane. (Physically, you could think of projecting the Earth onto a plane, with the South Pole sitting on the plane and the North Pole directly above.) This is one type of projection – your assignment this week is to investigate other types of projection.

Now imagine a two-dimensional version of this projection: A circle sitting on a line. For any point (other than the North Pole) on the surface (circle), the projection is obtained by constructing the line through this point and the North Pole, and then constructing the point where this line meets the plane (line) on which the surface (circle) sits.
Construct a diagram to illustrate the projection, but starting with an arbitrary point on the fixed line and constructing the point on the circle that projects onto it. A static version looks like Diagram 1 below:

![Diagram 1](image)

**Diagram 1**

**Inversion:**

2. Follow the construction process below to get the mirror effects.

   a. Start with a circle 1 centered at point S (which corresponds to the South Pole), and a point P randomly chosen in the sketch plane. Drag P so that it is to the right of S and a little above S so that your diagrams will be more easily compared to the ones shown below. We will now find a point that is P's mirror image by projecting P back onto a circle (as if we were finding the point
on Earth that projects onto the point P on the map), reflecting that point onto the opposite hemisphere, and then projecting that point back onto the plane ... Well, you'll see how it goes! Check back here after going through the details.

b. Construct the ray from S through P. (S is the ray's endpoint.)

c. Construct the point of intersection, T, of circle 1 with ray SP.

d. Mark S as a center, and rotate T by 90 degrees about S, producing point N.

e. Construct the segment NS, its midpoint O, and a circle 2 centered at O and passing through N. So far, the diagram should look something like this:
f. Construct the segment NP and its point of intersection, Q, with circle 2. (This locates the point on earth that corresponds to P.

g. Now to reflect to the other hemisphere: Construct the line through O that is perpendicular to NS, mark it as a mirror, and construct the reflection of Q, Q', in this new line.

h. Finally, project the point on Earth, Q', onto the map: Construct the ray NQ', with endpoint N, and its point of intersection, P', with the ray SP. Now the diagram should look something like this:
Diagram 3

Note that if your point $P$ is inside circle 1, $P'$ will be outside that circle, and $Q$ and $Q'$ will be in the southern and northern hemispheres, respectively.

The point $P'$ is called the **inversion of $P$ with respect to the circle 1**.

3. The final part of this lab is to investigate the mirror properties of inversion with the help of Sketchpad.

Start by hiding all those nice similar triangles and, actually, everything **except** the points $S$, $P$, ...
P', and the control point for circle 1, which you may have already hidden. To find this control point if it is hidden, just hide everything but S, P, and P', then Show All Hidden. You'll see the control point, which is the only object not connected to anything, except circle 1. Being careful not to click outside the selections, deselect the control point, and then hide everything selected again. Label the control point C.

Create a new tool: Select all four point and go to the Tool Bar and choose Create a New Tool by pressing the bottom and dragging it to choose Create New Tool. Click on Show Script View, title this script "Inversion," and press OK.

The script should appear on your monitor, with three objects as **Given** and about 13 **Steps** that construct other objects, eventually hiding them all, except P's inverse, P'.

Save your file (the sketch, that is), so that you can access this script; it will be available whenever you open this file.

Test how the script shows the funny mirror: Open a new sketch. You need to construct the givens, S, C, and P, but do it in the following
way to utilize Sketchpad's dynamic capability. Construct a circle and about twenty points outside the circle, so that they outline some interesting asymmetric shape, like the letter P. Select those twenty points, in the "right" order, so that segments formed by successive points would outline the shape you want. (Actually, this doesn't matter - you'll be able to drag these points wherever you want later to produce the shapes you want.) Choose Polygon Interior from the Construct menu, and then with this interior still selected, choose Point on Interior again from the Construct menu. Make this random point a color different from the other points, say blue, so that it can be distinguished from the others and selected as one of the givens.

Play the script, with the circle's center as S, the circle's control point as C, and the random interior point of the polygon as P. P's inversion, P', with respect to the circle should appear. Drag P around to see where P' goes.

Now construct the locus of P' as P ranges around the polygon: Select P, the polygon inertior, and P', and then choose Locus from the Construct menu. A lovely shape should appear: the image of whatever shape you made.
Experiment by dragging points on the polygon around to see what images you can produce.

Try dragging all of the polygon to the interior of the circle.
Consider the following diagram to construct the stereographic projection:
Stereographic projection is the mapping $P \mapsto P'$ from the sphere to the equatorial plane.

It has a number of important properties:

1. When $P$ lies on the equator, then $P = P'$ so the image of the equator is itself. More precisely, the equator is left fixed by the transformation $P \mapsto P'$. For convenience, let’s agree to call this circle the **equatorial circle**.

2. When $P$ lies in the Northern hemisphere then $P'$ lies inside the equatorial circle, while if $P$ lies in the Southern hemisphere, $P'$ lies outside the equatorial circle.

3. Since the ray passing through the South Pole and $P$ approaches the tangent line to the sphere at the South Pole, and so becomes parallel to the equatorial plane, as $P$ approaches the South Pole, the image of the South Pole under stereographic projection is identified with infinity in the equatorial plane.

4. There is a 1-1 correspondence between the equatorial plane and the set of all points on the sphere excluding the South Pole.

5. The image of any line of longitude, *i.e.*, any great circle passing through the North and South Poles, is a straight line passing through the center of the equatorial circle. Conversely, the pre-image of any straight line through the center of the equatorial circle is a line of longitude on the sphere.

6. The image of any line of latitude on the sphere is a circle in the equatorial plane concentric to the equatorial circle.

7. The image of any great circle on the sphere is a circle in the equatorial plane. Now every great circle intersects the equator at diametrically opposite points on the equator. On the other hand, the points on the equator are fixed by stereographic projection, so we see that the image of any great circle on the sphere is a circle in the equatorial plane passing through diametrically opposite points on the equatorial circle.

8. Stereographic projection is *conformal* in the sense that it preserves angle measure. In other words, if the angle between the tangents at the point of intersection of two great circles is $\theta$, then the angle between the tangents at the points of intersection of the images of these great circles is again $\theta$. 
**An algebraic approach:**

Let $\Sigma$ be a unit-sphere centered at the origin. Points on $\Sigma$ are described by:

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

and the equation of the equatorial plane be:

$$x^2 + y^2 = 1$$

Using similar triangles we can establish the mapping:

$\Pi: P(\alpha, \beta, \gamma) \rightarrow P'(x, y)$ by

$$x = \frac{\alpha}{1 + \gamma} \text{ and } y = \frac{\beta}{1 + \gamma} \{\text{an exercise}\}$$

Calculate what $x^2 + y^2 =$

and where the north, south poles and equatorial plane gets mapped to.